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On surface radiation conditions for high-frequency wave scattering

Andreas Atle[a,*], Bjorn Engquist[b]

[a] *Stockholm University, Stockholm, Sweden*
[b] *University of Texas at Austin, TX 78712, USA*

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Abstract

A new approximation of the logarithmic derivative of the Hankel function is derived and applied to high-frequency wave scattering. We re-derive the on surface radiation condition (OSRC) approximations that are well suited for a Dirichlet boundary in acoustics. These correspond to the Engquist–Majda absorbing boundary conditions. Inverse OSRC approximations are derived and they are used for Neumann boundary conditions. We obtain an implicit OSRC condition, where we need to solve a tridiagonal system. The OSRC approximations are well suited for moderate wave numbers. The approximation of the logarithmic derivative is also used for deriving a generalized physical optics approximation, both for Dirichlet and Neumann boundary conditions. We have obtained similar approximations in electromagnetics, for a perfect electric conductor. Numerical computations are done for different objects in 2D and 3D and for different wave numbers. The improvement over the standard physical optics is verified.

Keywords: Logarithmic derivative; On surface radiation condition; Physical optics; Acoustics; Electromagnetics

1. Introduction

In high-frequency wave scattering, computations of the potential of a scatterer surface lead to a full linear system of equations where accuracy requires the number of unknowns to increase with frequency. There exist various techniques to speedup the computation of the solution of these linear system of equations, e.g., iterative solvers and fast multipole methods. Another alternative to accelerate the computation is to use on surface radiation condition (OSRC) where the potential is obtained without solving any system of equations. Originally, OSRC was derived by using absorbing boundary conditions (ABC) directly on the scattering surface in order to obtain the potential in the incoming wave. This, however, only applies to a convex scatterer. An approximation of the potential is obtained without solving any system of equations. Some of the first OSRCs was derived from the Engquist–Majda [12,11] and the Bayliss–Turkel [8,7] ABCs. Generalizations of the OSRCs arising from ABCs have been derived using micro-local analysis [6,5]. In the case of a general non-convex scatterer, the OSRC can be combined with a full linear system or basic physical optics (PO) for the non-convex parts of the scatterer.

* Corresponding author.
E-mail addresses: atle@nada.kth.se (A. Atle), engquist@math.utexas.edu (B. Engquist).
URL: http://www.nada.kth.se/~atle (A. Atle).

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Here we consider a third approach to derive an OSRC. If we assume that the curvature is locally constant, we can use the analytical solution on a circular cylinder (in 2D) or a sphere (in 3D) and obtain an approximation of the logarithmic derivatives of the Hankel functions. This type of approximation was derived by Alpert et al. [2], and yields a rational approximation. Their expansion is not applicable in the context of OSRC, since there are different coefficients in the rational approximation for different orders of the Hankel functions. Another rational approximation well suited for the OSRC was developed by Calvo et al. [10].

In this article, we approximate the logarithmic derivative of the Hankel function by an expression which has unique Taylor and Padé expansions. This expression can be used to re-derive the OSRC obtained from the Engquist–Majda ABC. By using the Padé expansion we obtain the inverse OSRC operator, which is useful for Neumann problems. We also use the Taylor expansion to obtain a generalization to PO and we show how to generalize the OSRC and PO approximations to electromagnetic problems with an electric conductor.

2. Logarithmic derivatives of the Hankel function

The analytical expression for the potential $\partial_n u^{\mathrm{sc}}$ on a circular cylinder with radius R induced by the incoming field $u^{\mathrm{inc}}(r,\phi)=\mathrm{e}^{-\mathrm{i}kr\cos\phi}$ can be written in the Fourier-dual of ϕ. For a sphere we consider the incoming field $u^{\mathrm{inc}}(r,\theta,\phi)=\mathrm{e}^{-\mathrm{i}kr\cos\theta}$ and the analytical expression in the corresponding Fourier-dual in θ can be written as

$$\partial_n \hat{u}_m^{\mathrm{sc}} = \begin{cases} (\partial_r \log H_m^{(1)}(kr)|_{r=R})\hat{u}_m^{\mathrm{sc}} & \text{on a circular cylinder,} \\ (\partial_r \log h_m^{(1)}(kr)|_{r=R})\hat{u}_m^{\mathrm{sc}} & \text{on a sphere,} \end{cases}$$

where $H_m^{(1)}$ and $h_m^{(1)}$ are the Hankel function and spherical Hankel function of order m, respectively.

Approximating the logarithmic derivative with a rational function in the dual variable m corresponds to taking certain surface derivatives in physical space.

Let

$$S_m(x)=\frac{\pi x}{2}\,|H_m^{(1)}(x)|^2,$$

then the following expressions can be derived:

$$\partial_x \log H_m^{(1)}(x) = -\frac{1}{2x}+\frac{\partial_x S_m(x)}{2S_m(x)}+\mathrm{i}\,\frac{1}{S_m(x)},$$

$$\partial_x \log h_{m-1/2}^{(1)}(x) = -\frac{1}{x}+\frac{\partial_x S_m(x)}{2S_m(x)}+\mathrm{i}\,\frac{1}{S_m(x)}.$$

The function S_m has the expansion, see [1],

$$S_m(x)=1+\sum_{p=1}^{\infty} a_p(m)\left(\frac{m}{x}\right)^{2p}, \quad a_p(m)=\frac{(2p-1)!!}{(2p)!!}\prod_{q=1}^{p}\left(1-\frac{(q-1/2)^2}{m^2}\right).$$

If we keep $y=m/x$ constant, we get a second-order approximation, $\tilde{S}_m(x)$, of $S_m(x)$ in x, if the factors in the coefficients $a_p(m)$ are approximated by one. The approximation $\tilde{S}_m$ can be written as a closed expression

$$\tilde{S}_m(x)=\frac{1}{\sqrt{1-m^2/x^2}}. \tag{1}$$

In Fig. 1, we present the relative error in the approximation (1) for $x=5$, 20, 80 and 320. It clearly shows that the relative error decreases when x is increased and the numerical order $\log_2((\tilde{S}_m(x)-S_m(x))/(\tilde{S}_{2m}(2x)-S_{2m}(2x)))\to 2$ as x increases.

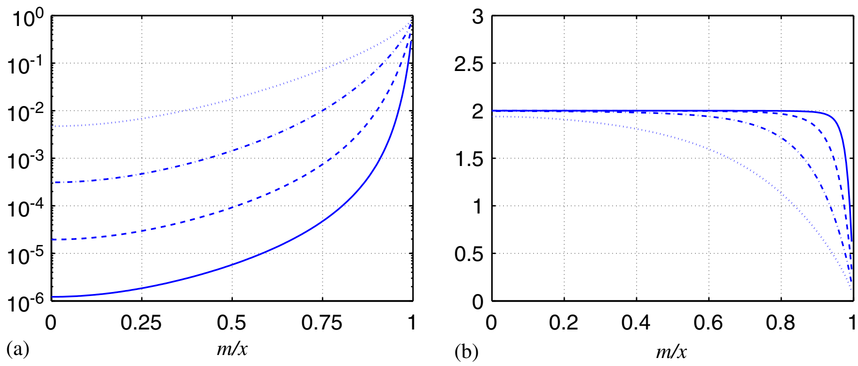

Fig. 1. Approximation of $S_m(x)$ for $x = 5$(..), 20(-.), 80(--) and 320(-): (a) relative error in approximation of $S_m(x)$; (b) order of approximation of $S_m(x)$.

If we use the approximation, $\tilde{S}$ in 1, the expressions for the logarithmic derivatives can now be computed, with $x = kR$,

$$\partial_r \log H_m^{(1)}(kr)|_{r=R} = -\frac{1}{2R}\,\frac{1}{1-m^2/k^2R^2} + \mathrm{i}k\sqrt{1-\frac{m^2}{k^2R^2}} + \mathcal{O}\left(\frac{1}{kR^2}\right), \tag{2}$$

$$\partial_r \log h_{m-1/2}^{(1)}(kr)|_{r=R} = -\frac{1}{2R}\,\frac{2-m^2/k^2R^2}{1-m^2/k^2R^2} + \mathrm{i}k\sqrt{1-\frac{m^2}{k^2R^2}} + \mathcal{O}\left(\frac{1}{kR^2}\right), \tag{3}$$

where the order is obtained when m/kR is kept constant.

A significant difference between this approximation and the approximation by Alpert et al. [2] is that the same approximations (2)–(3) hold for essentially all values of the dual parameter m in the interesting region $m/kR < 1$. This implies that we can identify the dual parameter m with surface derivatives.

3. Derivation of OSRCs

In the derivation of the original OSRC, Kriegsmann et al. [13] proposed that the cylindrical surface should locally be replaced by a segment of its osculating circle. They suggested the following substitution for the OSRC:

$$\partial_r \to \partial_n, \quad \frac{1}{R} \to \kappa, \quad -\frac{m^2}{R^2} \to \partial_s^2,$$

where κ is the curvature and s is the arclength parameterization of the surface. In 3D the corresponding substitution is valid:

$$\partial_r \to \partial_n, \quad \frac{1}{R} \to \mathscr{H}, \quad -\frac{m(m+1)}{R^2} \to \Delta_\Gamma,$$

where $\mathscr{H}$ is the mean curvature and Δ_Γ is the surface Laplacian. In our construction of the OSRC we use a pseudo-differential operator such that

$$\partial_n u^{\mathrm{sc}} = kF^{\mathrm{2D}}\left(\frac{k}{\kappa}, \frac{1}{k^2}\partial_s^2\right)u^{\mathrm{sc}}, \quad F^{\mathrm{2D}}(x,y) = -\frac{1}{2x}\,\frac{1}{1+y} + \mathrm{i}\sqrt{1+y}, \tag{4}$$

$$\partial_n u^{\mathrm{sc}} = kF^{\mathrm{3D}}\left(\frac{k}{\mathscr{H}}, \frac{1}{k^2}\Delta_\Gamma - \frac{\mathscr{H}^2}{4k^2}\right)u^{\mathrm{sc}}, \quad F^{\mathrm{3D}}(x,y) = -\frac{1}{2x}\,\frac{2+y}{1+y} + \mathrm{i}\sqrt{1+y}. \tag{5}$$

The different OSRCs can now be constructed by taking different Padé(i/j) approximations $F_{i/j}^{2D/3D}$ of $F^{2D/3D}$ where

$$F_{i/j}(x, y) = \frac{\alpha_0(x) + \alpha_1(x)y + \cdots + \alpha_i(x)y^i}{\beta_0(x) + \beta_1(x)y + \cdots + \beta_j(x)y^j}.$$

The discrepancy between $m(m+1)$ and $(m+\frac{1}{2})^2$ introduces the correction $-\mathscr{H}^2/4k^2$ to the surface gradient, but can be omitted for approximations up to second order. The OSRC suited for Dirichlet conditions is derived using Padé(1/0), while the conditions for Neumann problems uses Padé(0/1),

$$u_n^{\mathrm{sc}} = \left(\mathrm{i}k - \frac{\kappa}{2}\right) u^{\mathrm{sc}} + \frac{1}{2k^2}(\mathrm{i}k + \kappa)u_{ss}^{\mathrm{sc}} \quad \text{2D, Dirichlet,} \tag{6}$$

$$u_n^{\mathrm{sc}} = (\mathrm{i}k - \mathscr{H})u^{\mathrm{sc}} + \frac{1}{2k^2}(\mathrm{i}k + \mathscr{H})\Delta_\Gamma u^{\mathrm{sc}} \quad \text{3D, Dirichlet,} \tag{7}$$

$$\left(\mathrm{i}k - \frac{\kappa}{2}\right)^2 u^{\mathrm{sc}} = \left(\mathrm{i}k - \frac{\kappa}{2}\right) u_n^{\mathrm{sc}} - \frac{1}{2k^2}(\mathrm{i}k + \kappa)u_{nss}^{\mathrm{sc}} \quad \text{2D, Neumann,} \tag{8}$$

$$(\mathrm{i}k - \mathscr{H})^2 u^{\mathrm{sc}} = (\mathrm{i}k - \mathscr{H})u_n^{\mathrm{sc}} - \frac{1}{2k^2}(\mathrm{i}k + \mathscr{H})\Delta_\Gamma u_n^{\mathrm{sc}} \quad \text{3D, Neumann.} \tag{9}$$

The 2D Dirichlet OSRC coincides with the Engquist–Majda ABC in cylindrical coordinates [11]. The Neumann OSRCs are new versions, not seen in the literature. These OSRCs yield an explicit formula also for a Neumann problem, which earlier required the solution of a sparse matrix system. Instead by using Padé(1/1), we obtain implicit OSRCs:

$$B_0^{2D}u_n^{\mathrm{sc}} + B_2^{2D}u_{nss}^{\mathrm{sc}} = A_0^{2D}u^{\mathrm{sc}} + A_2^{2D}u_{ss}^{\mathrm{sc}} \quad \text{2D, implicit,} \tag{10}$$

$$B_0^{3D}u_n^{\mathrm{sc}} + B_2^{3D}\Delta_\Gamma u_n^{\mathrm{sc}} = A_0^{3D}u^{\mathrm{sc}} + A_2^{3D}\Delta_\Gamma u^{\mathrm{sc}} \quad \text{3D, implicit} \tag{11}$$

with the coefficients

$$B_0^{2D} = 2\mathrm{i}k + 2\kappa, \quad B_2^{2D} = \frac{1}{2k^2}(\mathrm{i}k + 4\kappa),$$

$$A_0^{2D} = -(\kappa^2 - \mathrm{i}k\kappa + 2k^2), \quad A_2^{2D} = -\left(\frac{3}{2} - \frac{15\mathrm{i}\kappa}{4k}\right),$$

$$B_0^{3D} = \mathrm{i}k + \mathscr{H}, \quad B_2^{3D} = \frac{1}{4k^2}(\mathrm{i}k + 4\kappa),$$

$$A_0^{3D} = -(\mathscr{H}^2 + k^2), \quad A_2^{3D} = -\left(\frac{3}{4} - \frac{7\mathrm{i}\mathscr{H}}{4k} + \frac{\mathscr{H}^2}{2k^2}\right).$$

4. Generalization of PO

By taking into account the curvature, the PO approximation can be generalized. The original PO is exact for plane waves illuminating an infinite plane and can be written as

$$u_n^{\mathrm{tot}} = \begin{cases} 2u_n^{\mathrm{inc}} & \text{in } \Gamma_{\mathrm{lit}}, \\ 0 & \text{in } \Gamma_{\mathrm{shad}}, \end{cases} \quad \text{Dirichlet case,} \tag{12}$$

$$u^{\mathrm{tot}} = \begin{cases} 2u^{\mathrm{inc}} & \text{in } \Gamma_{\mathrm{lit}}, \\ 0 & \text{in } \Gamma_{\mathrm{shad}}, \end{cases} \quad \text{Neumann case,} \tag{13}$$

where Γ_{lit} is the illuminated region of the scatterer and Γ_{shad} is the shadow region. These conditions are the same in 2D and in 3D. The shadow region for a general object can be computed by geometrical optics ray tracing or by using a levelset technique introduced in [14]. A generalization of PO can be obtained by the following steps:

- Use a plane wave ansatz.
- Find the Taylor expansion of the pseudo-differential operator (4) in 2D or (5) in 3D.
- Compute surface derivatives of the incoming plane wave and insert the surface derivatives in the Taylor expansion.
- Find the new closed formula for this Taylor expansion.

The circular cylinder case in 2D, with an incoming plane wave along the negative x-axis $u^{\text{inc}} = \mathrm{e}^{-\mathrm{i}kx} = \mathrm{e}^{-\mathrm{i}kr\cos\phi}$, admits the following Taylor expansion of (4),

$$u_n^{\text{sc}} = \sum_{p=0}^{\infty} \left(-\frac{\kappa}{2}(-1)^p + \mathrm{i}k \binom{1/2}{p} \right) \frac{\partial_s^{2p} u^{\text{sc}}}{k^{2p}}, \tag{14}$$

where the surface derivatives of the incoming plane wave are

$$\frac{\partial_s^{2p} u^{\text{inc}}}{k^{2p}} = \left((-\sin^2(\phi))^p + \frac{\mathrm{i}\kappa}{k}\cos(\phi)(2p^2 - p)(-\sin^2(\phi))^{p-1} + \mathcal{O}(k^{-2}) \right) u^{\text{inc}}. \tag{15}$$

By using the Dirichlet boundary condition, $u^{\text{sc}} = -u^{\text{inc}}$ and inserting (15) in (14) we obtain

$$u_n^{\text{sc}} = -\mathrm{i}k|\cos\phi| u^{\text{inc}} + \delta_{\text{lit}} \frac{\kappa}{\cos^2\phi} u^{\text{inc}}, \quad \delta_{\text{lit}} = \frac{1}{2}(1 + \operatorname{sgn}\cos\phi) = \begin{cases} 1 & \text{on } \Gamma_{\text{lit}}, \\ 0 & \text{on } \Gamma_{\text{shad}}. \end{cases} \tag{16}$$

This expression can be generalized to arbitrary incoming plane waves $u^{\text{inc}}(\mathbf{x}) = \mathrm{e}^{\mathrm{i}\hat{\mathbf{m}}\cdot\mathbf{x}}$ by identifying $\cos\phi = -\hat{\mathbf{m}}\cdot\hat{\mathbf{n}}$ and $\sin\phi = |\hat{\mathbf{m}} \times \hat{\mathbf{n}}|$. The expression in (16) is unbounded as $\phi \to 90°$, i.e., at the grazing angle. We propose the correction $\cos^2\phi \to \cos^2\phi + (\delta\kappa/k)\sin^2\phi$ to avoid the singularity. The parameter δ can be chosen in the order of 1. The final generalization of PO for a Dirichlet cylinder is

$$u_n^{\text{tot}} = \begin{cases} 2u_n^{\text{inc}} + \dfrac{\kappa}{|\hat{\mathbf{n}}\cdot\hat{\mathbf{m}}|^2 + (\delta\kappa/k)|\hat{\mathbf{n}}\times\hat{\mathbf{m}}|^2} u^{\text{inc}} & \text{on } \Gamma_{\text{lit}}, \\ 0 & \text{on } \Gamma_{\text{shad}}. \end{cases} \tag{17}$$

In order to derive the corresponding PO condition for a Neumann cylinder we need an approximation of the inverse operator $G^{2\text{D}} = (F^{2\text{D}})^{-1}$. A second-order approximation $G^{2\text{D}}$ in x can be derived,

$$G^{2\text{D}}(x, y) = -\frac{1}{2x(1+y)^2} + \frac{1}{\mathrm{i}\sqrt{1+y}}, \tag{18}$$

for which the product

$$F^{2\text{D}}G^{2\text{D}} = 1 + \frac{1}{4x^2(1+y)^3}$$

is second order in x. Using the Taylor expansion of $G^{2\text{D}}$ yields

$$u^{\text{sc}} = \sum_{p=0}^{\infty} \left(-\frac{1}{2x}(p+1)(-1)^p - \mathrm{i}\binom{-1/2}{n} \right) \frac{\partial_s^{2p} u_n^{\text{sc}}}{k^{2p}}, \tag{19}$$

and the surface derivatives can be computed,

$$\frac{\partial_s^{2p} u_n^{\text{inc}}}{k^{2p}} = z^p u_n^{\text{inc}} + \kappa(2p^2 + p)z^p u^{\text{inc}} + \kappa(2p^2 - p)z^{p-1} u^{\text{inc}}, \quad z = -\sin^2\phi.$$

By computing the sum in (19), we obtain the closed formula for the potential with the correction $\cos^4\phi \to \cos^4\phi + (\delta\kappa/k)(1-\cos^4\phi)$ to avoid a singularity at the grazing angle $\phi = 90°$. The scaling with κ/k is needed in order to keep the second-order accuracy. The Neumann PO condition is

$$u^{\text{tot}} = \begin{cases} 2u^{\text{inc}} + \dfrac{\kappa}{k^2(|\hat{\mathbf{n}}\cdot\hat{\mathbf{m}}|^4 + (\delta\kappa/k)(1-|\hat{\mathbf{n}}\cdot\hat{\mathbf{m}}|^4))} u_n^{\text{inc}} & \text{on } \Gamma_{\text{lit}}, \\ 0 & \text{on } \Gamma_{\text{shad}}. \end{cases} \tag{20}$$

In 3D, the derivations are similar to the 2D case and we obtain the Dirichlet PO condition,

$$u_n^{\text{tot}} = \begin{cases} 2u_n^{\text{inc}} + \dfrac{\mathscr{H}(1+|\hat{\mathbf{n}}\cdot\hat{\mathbf{m}}|^2)}{|\hat{\mathbf{n}}\cdot\hat{\mathbf{m}}|^2 + (\delta\mathscr{H}/k)|\hat{\mathbf{n}}\times\hat{\mathbf{m}}|^2} u^{\text{inc}} & \text{on } \Gamma_{\text{lit}}, \\ 0 & \text{on } \Gamma_{\text{shad}}. \end{cases} \tag{21}$$

The Neumann PO condition in 3D is

$$u^{\text{tot}} = \begin{cases} 2u^{\text{inc}} + \dfrac{\mathscr{H}(1+|\hat{\mathbf{n}}\cdot\hat{\mathbf{m}}|^2)}{k^2(|\hat{\mathbf{n}}\cdot\hat{\mathbf{m}}|^4 + (\delta\mathscr{H}/k)(1-|\hat{\mathbf{n}}\cdot\hat{\mathbf{m}}|^4))} u_n^{\text{inc}} & \text{on } \Gamma_{\text{lit}}, \\ 0 & \text{on } \Gamma_{\text{shad}}, \end{cases} \tag{22}$$

where δ is a parameter and $\mathscr{H}$ is the mean curvature.

5. Electromagnetic case with an electric conductor

In electromagnetic scattering problems for an electric conductor, the main task is to compute the surface current $\mathbf{J}_{\text{s}}$ on the scatterer, for a given incoming field. As a model problem, the incoming field is $\mathbf{E}^{\text{inc}} = \mathbf{x}\mathrm{e}^{-\mathrm{i}kz}$ and $\mathbf{H}^{\text{inc}} = -Y\mathbf{y}\mathrm{e}^{-\mathrm{i}kz}$, where $Y = Z^{-1}$.

Ammari and Nedelec [4] introduce the electromagnetic operator, T, that expresses $\mathbf{J}_{\text{s}}$ in terms of the incoming wave,

$$\mathbf{J}_{\text{s}} = \hat{\mathbf{n}} \times \mathbf{H}^{\text{inc}} + T[\mathbf{E}_{\text{tan}}^{\text{inc}}] \quad \text{where } T[\mathbf{E}_{\text{tan}}^{\text{sc}}] = -\hat{\mathbf{n}} \times \mathbf{H}^{\text{sc}} = -\mathbf{J}_{\text{s}}^{\text{sc}}. \tag{23}$$

If we decompose $\mathbf{E}_{\text{tan}}^{\text{sc}}$ in a divergence part $\mathbf{E}_{\text{div}}^{\text{sc}}$ with basis $\mathbf{G}_l^m$ and a curl part $\mathbf{E}_{\text{curl}}^{\text{sc}}$ with basis $\mathbf{R}_l^m$,

$$\mathbf{E}_{\text{tan}}^{\text{sc}} = \mathbf{E}_{\text{div}}^{\text{sc}} + \mathbf{E}_{\text{curl}}^{\text{sc}}, \quad \mathbf{E}_{\text{div}}^{\text{sc}} = \sum_{l=1}^{\infty}\sum_{m=-l}^{l} \alpha_l^m \mathbf{G}_l^m, \quad \mathbf{E}_{\text{curl}}^{\text{sc}} = \sum_{l=1}^{\infty}\sum_{m=-l}^{l} \beta_l^m \mathbf{R}_l^m,$$

then the electromagnetic operator is

$$ZT[\mathbf{E}_{\text{tan}}^{\text{sc}}] = \sum_{l=1}^{\infty}\sum_{m=-l}^{l} \frac{1}{\gamma_l}\alpha_l^m \mathbf{G}_l^m + \gamma_l \beta_l^m \mathbf{R}_l^m \quad \text{with } \gamma_l = \frac{1 + R\partial_n \log h_l^{(1)}(kR)}{\mathrm{i}kR}.$$

In [3] the surface current is expressed in surface derivatives of the incoming electric field. The OSRC presented there is

$$Z\mathbf{J}_{\text{s}}^{\text{sc}} = \mathbf{E}_{\text{tan}}^{\text{inc}} - \frac{1}{2k^2}(\mathbf{grad}_\Gamma \operatorname{div}_\Gamma \mathbf{E}_{\text{tan}}^{\text{inc}} + \mathbf{curl}_\Gamma \operatorname{curl}_\Gamma \mathbf{E}_{\text{tan}}^{\text{inc}}). \tag{24}$$

In this section we will improve the OSRC (24). We will also derive a generalization of the PO for an electric conductor, using the same technique as in the scalar case. Expression (5) yields the approximation

$$\gamma_l = \sqrt{1+y} + \frac{1}{2\mathrm{i}kR}\frac{y}{1+y} \quad \text{where } y = -\frac{(l+1/2)^2}{k^2R^2}. \tag{25}$$

In order to derive an OSRC condition, we can identify the parameter l with surface derivatives by

$$\mathbf{grad}_\Gamma \operatorname{div}_\Gamma \mathbf{E}^{\mathrm{sc}}_{\tan} = -\frac{l(l+1)}{R^2}\mathbf{E}^{\mathrm{sc}}_{\mathrm{div}}, \quad -\mathbf{curl}_\Gamma \operatorname{curl}_\Gamma \mathbf{E}^{\mathrm{sc}}_{\tan} = -\frac{l(l+1)}{R^2}\mathbf{E}^{\mathrm{sc}}_{\mathrm{curl}}. \tag{26}$$

If we take the Padé(1/0) expansion of γ_l for the rotational part and Padé(0/1) expansion of γ_l for the gradient part of the field, we obtain a new OSRC for an electric conductor,

$$Z\mathbf{J}^{\mathrm{sc}}_{\mathrm{s}} = \mathbf{E}^{\mathrm{inc}}_{\tan} + \frac{\mathrm{i} - kR}{2k^3R}(\mathbf{grad}_\Gamma \operatorname{div}_\Gamma \mathbf{E}^{\mathrm{inc}}_{\tan} + \mathbf{curl}_\Gamma \operatorname{curl}_\Gamma \mathbf{E}^{\mathrm{inc}}_{\tan}). \tag{27}$$

Remark 1. The difference between $(l+1/2)^2$ in (25) and $l(l+1)$ in (26) produces an error which is second order in k and can therefore be neglected.

In order to develop a PO approximation for the electromagnetic case we need an explicit expression for $1/\gamma_l$. We write

$$\gamma_l = F^{\mathrm{EM}}(x, y) = \sqrt{1+y} - \mathrm{i}\frac{1}{2x}\frac{y}{1+y} + \mathcal{O}\left(\frac{1}{x^2}\right), \tag{28}$$

$$\frac{1}{\gamma_l} = G^{\mathrm{EM}}(x, y) = \frac{1}{\sqrt{1+y}} + \mathrm{i}\frac{1}{2x}\frac{y}{(1+y)^2} + \mathcal{O}\left(\frac{1}{x^2}\right), \tag{29}$$

where $x = kR$ and $y = -(l+1/2)^2/x^2$. Observing that the first term in the Taylor-expansions of F^{EM} and G^{EM} equals one gives

$$F^{\mathrm{EM}}\left(kR, -\frac{\mathbf{curl}_\Gamma \operatorname{curl}_\Gamma}{k^2}\right)\mathbf{E}_{\tan} = \mathbf{E}_{\mathrm{div}} + F^{\mathrm{EM}}\left(kR, -\frac{\mathbf{curl}_\Gamma \operatorname{curl}_\Gamma}{k^2}\right)\mathbf{E}_{\mathrm{curl}}, \tag{30}$$

$$G^{\mathrm{EM}}\left(kR, \frac{\mathbf{grad}_\Gamma \operatorname{div}_\Gamma}{k^2}\right)\mathbf{E}_{\tan} = \mathbf{E}_{\mathrm{curl}} + G^{\mathrm{EM}}\left(kR, \frac{\mathbf{grad}_\Gamma \operatorname{div}_\Gamma}{k^2}\right)\mathbf{E}_{\mathrm{div}}. \tag{31}$$

Combining (30) and (31) with the electromagnetic operator (23) yields the following equation for the surface current:

$$Z\mathbf{J}^{\mathrm{sc}}_{\mathrm{s}} = -\mathbf{E}^{\mathrm{inc}}_{\tan} + F^{\mathrm{EM}}\left(kR, -\frac{\mathbf{curl}_\Gamma \operatorname{curl}_\Gamma}{k^2}\right)\mathbf{E}^{\mathrm{inc}}_{\tan} + G^{\mathrm{EM}}\left(kR, \frac{\mathbf{grad}_\Gamma \operatorname{div}_\Gamma}{k^2}\right)\mathbf{E}^{\mathrm{inc}}_{\tan}. \tag{32}$$

The high-order surface derivatives of the incoming field can be computed in the following way:

$$\begin{aligned}
\frac{(-\mathbf{curl}_\Gamma \operatorname{curl}_\Gamma)^p \mathbf{E}_{\tan}}{k^{2p}} &= -\left(\frac{1}{\mathrm{i}kR\cos\theta}(-\sin^2\theta)^{p-1}\right)E_\theta\hat{\theta} \\
&\quad + \left((-\sin^2\theta)^p - \frac{(2p^2-1)\cos\theta}{\mathrm{i}kR}(-\sin^2\theta)^{p-1}\right)E_\phi\hat{\phi}, \\
\frac{(\mathbf{grad}_\Gamma \operatorname{div}_\Gamma)^p \mathbf{E}_{\tan}}{k^{2p}} &= \left((-\sin^2\theta)^p + \frac{(2p+1)}{\mathrm{i}kR\cos\theta}(-\sin^2\theta)^{p-1} - \frac{2\cos\theta(p^2+p)}{\mathrm{i}kR}(-\sin^2\theta)^{p-1}\right)E_\theta\hat{\theta} \\
&\quad - \left(\frac{\cos\theta}{\mathrm{i}kR}(-\sin^2\theta)^{p-1}\right)E_\phi\hat{\phi}.
\end{aligned}$$

If the high-order derivatives are inserted in the Taylor expansion of (32), a second-order PO approximation is obtained,

$$\mathbf{J}^{\mathrm{sc}}_{\mathrm{s}} = \left(\frac{1}{|\cos\theta|} + \frac{\delta_{\mathrm{lit}}\sin^2\theta}{\mathrm{i}kR\cos^4\theta}\right)E^{\mathrm{inc}}_\theta\hat{\theta} + \left(|\cos\theta| - \frac{\delta_{\mathrm{lit}}\sin^2\theta}{\mathrm{i}kR\cos^2\theta}\right)E^{\mathrm{inc}}_\phi\hat{\phi},$$

and, by identifying

$$H_\phi^{\text{inc}} = -Y \frac{1}{\cos\theta} E_\theta^{\text{inc}}, \quad H_\theta^{\text{inc}} = Y \cos\theta E_\phi^{\text{inc}},$$

we obtain a representation in $\mathbf{H}$, which we state in Theorem 1.

Theorem 1. *Consider an incoming field given by* $\mathbf{E}^{\text{inc}} = \hat{\mathbf{x}}\mathrm{e}^{-\mathrm{i}kz}$ *and* $\mathbf{H}^{\text{inc}} = -Y\hat{\mathbf{y}}\mathrm{e}^{-\mathrm{i}kz}$. *A second-order PO approximation admits the formula*

$$\mathbf{J}_\mathrm{s}^{\text{tot}} = \begin{cases} 2\hat{\mathbf{n}} \times \mathbf{H}^{\text{inc}} - \dfrac{1}{\mathrm{i}kR} \dfrac{\sin^2\theta}{\cos^3\theta + (\delta/kR)(1-\cos^3\theta)} (H_\phi^{\text{inc}}\hat{\theta} + H_\theta^{\text{inc}}\hat{\phi}) & \text{in } \Gamma_{\text{lit}}, \\ 0 & \text{in } \Gamma_{\text{shad}}, \end{cases} \tag{33}$$

where δ *is a parameter in the order of* 1.

6. Numerical computations

When our different approximations are used in computational acoustics, there are at least two different kinds of errors. The first error is related to the approximations of the exact expressions, when the object is a circular cylinder or a sphere. A second error is introduced when the curvature is varying. We present computations on simple objects for all different approximations in this paper. The errors introduced from the varying curvature will be presented in 2D, for an ellipse in Fig. 7. The different PO approximations coincide in the shadow region Γ_{shad}. The parameter in the PO approximations (17), (20)–(22) is chosen to $\delta = 4$ in the numerical computations. In Fig. 2 we present the relative error in the potential for a circular cylinder with Dirichlet boundary conditions. The implicit OSRC gives the smallest relative error when $kR = 20$. In the case where $kR = 80$ the PO approximations yield better results. In Fig. 3 we present the relative error for a circular cylinder with Neumann boundary conditions. The implicit OSRC is better for Neumann than for Dirichlet boundary conditions when $kR = 80$. The PO approximations become better with increasing wave numbers while the second-order region in the OSRC-approximations becomes narrower. In Figs. 4 and 5 we present the relative errors of the potential when the scatterer is a sphere. The results are almost identical to the circular cylinder case. In Fig. 6, we present the different approximations obtained in the electromagnetic case, with a perfectly electric conductor.

To see what happens with varying curvature, we do numerical computations of an elliptic cylinder with horizontal axis $a = 3$ and vertical axis $b = 1$. The angle of incidence is 45°. As a reference solution we use a collocation-based 2D integral equation solver for a Dirichlet scatterer. An alternative is to express the exact solution in terms of Mathieu functions [9]. The results are presented in Fig. 7, for the case $k = 10$. In Fig. 8, we present the relative error of the two different OSRCs (24) and (27), for electromagnetic current, when $kR = 10$. As the wave number increases, the difference between the two approximations becomes smaller. The new OSRC (27) appears to be better than (24) for low to moderate kR.

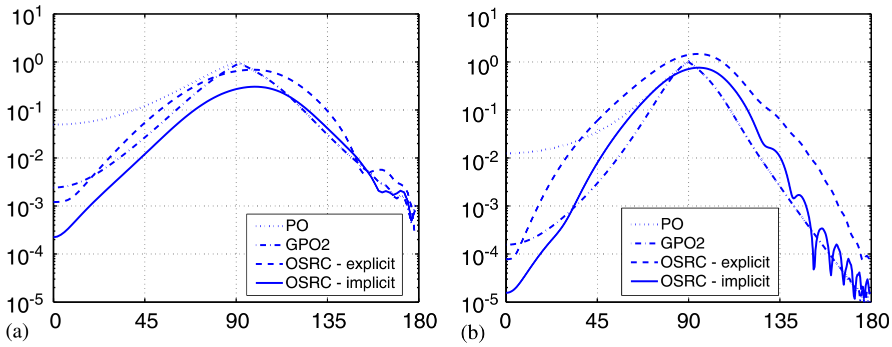

Fig. 2. Relative error in potential for a circular cylinder with Dirichlet boundary conditions, using PO (12), GPO2 (17), OSRC-explicit (6) and OSRC-implicit (10): (a) $kR = 20$; (b) $kR = 80$.

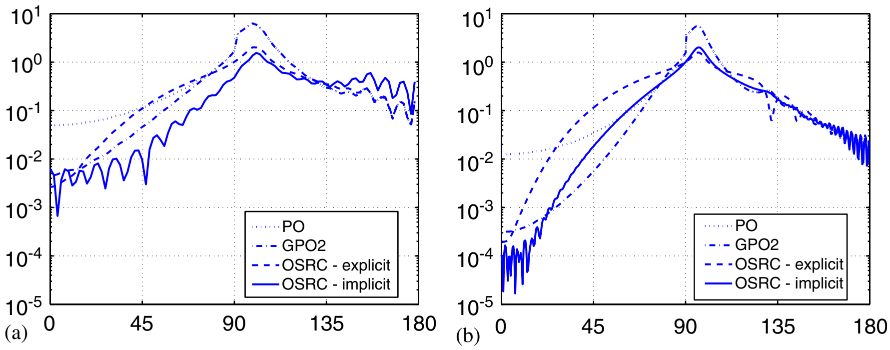

Fig. 3. Relative error in potential for a circular cylinder with Neumann boundary conditions, using PO (13), GPO2 (20), OSRC-explicit (8) and OSRC-implicit (10): (a) $kR = 20$; (b) $kR = 80$.

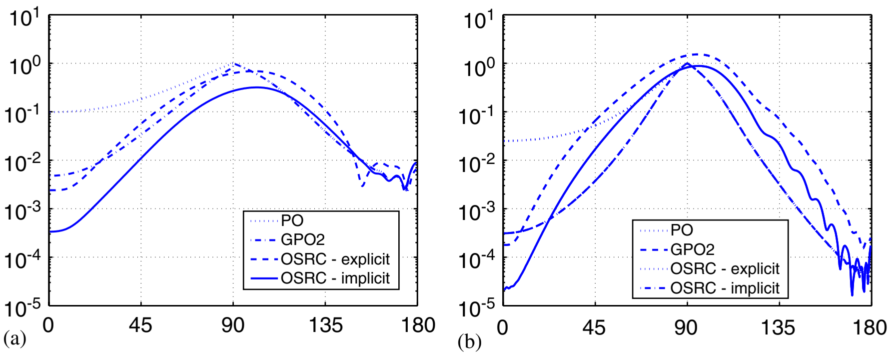

Fig. 4. Relative error in potential for a sphere with Dirichlet boundary conditions, using PO (12), GPO2 (21), OSRC-explicit (7) and OSRC-implicit (11): (a) $kR = 20$; (b) $kR = 80$.

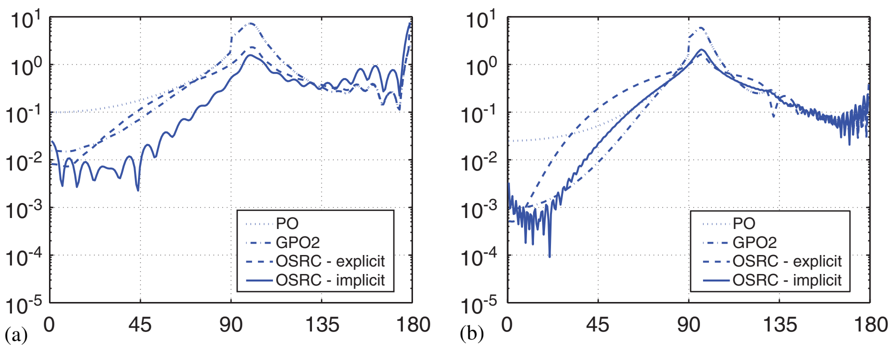

Fig. 5. Relative error in potential for a sphere with Neumann boundary conditions, using PO (12), GPO2 (22), OSRC-explicit (9) and OSRC-implicit (11): (a) $kR = 20$; (b) $kR = 80$.

7. Summary and conclusions

Several different OSRC and PO approximations have been presented, by using approximations of the logarithmic derivative of different Hankel functions. The approximations are second order in $(kR)^{-1}$, when the quotient m/kR is kept constant and where R is the radius of curvature. Different Padé expansions give different OSRC approximations.

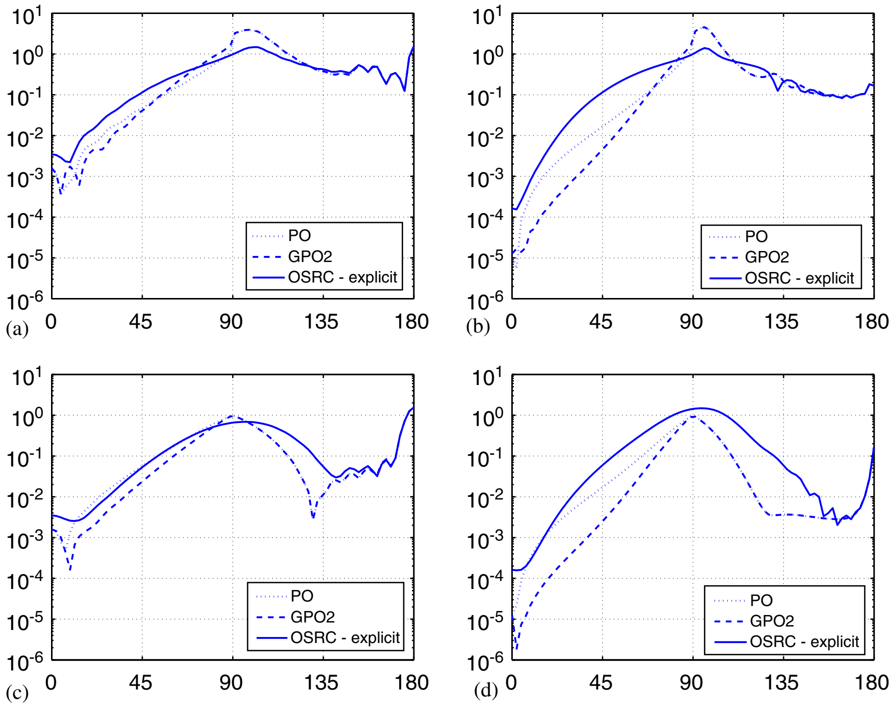

Fig. 6. Relative error in current for a sphere with perfectly conducting boundary conditions, using PO ($\mathbf{J}^{sc} = \hat{\mathbf{n}} \times \mathbf{H}^{inc}$), GPO2 (33) and OSRC-explicit (27): (a) $\phi = 0°$, $kR = 20$; (b) $\phi = 0°$, $kR = 80$; (c) $\phi = 90°$, $kR = 20$; and (d) $\phi = 90°$, $kR = 80$.

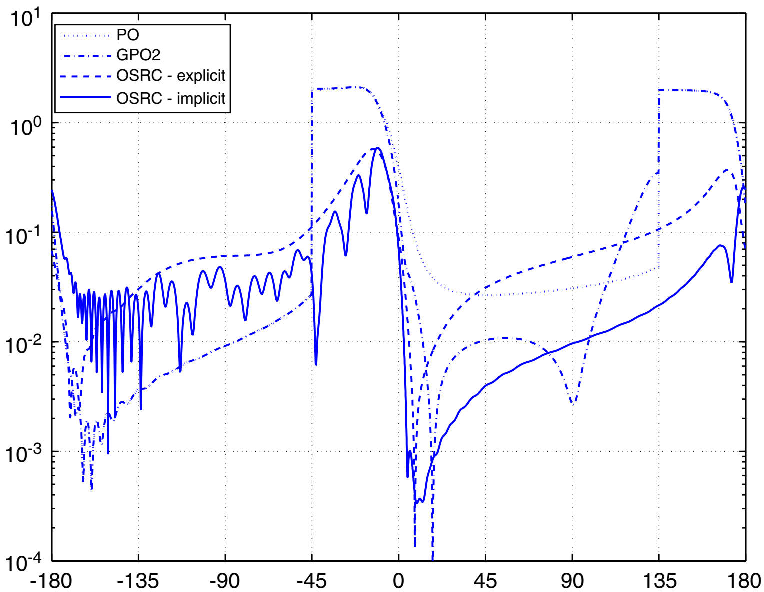

Fig. 7. Relative error in potential for a elliptic cylinder, with the horizontal axis $a = 3$ and the vertical axis $b = 1$ with Dirichlet boundary conditions. The wave number $k = 10$ and the angle of incidence is 45°.

Approximation of all terms in the Taylor expansion of the expression for the logarithmic derivative yields a generalized PO approximation, which is second order in $(kR)^{-1}$ and thus a substantial improvement over the standard PO method. The OSRC approximations have a smaller error when the wave number is moderate. For increasing wave numbers the

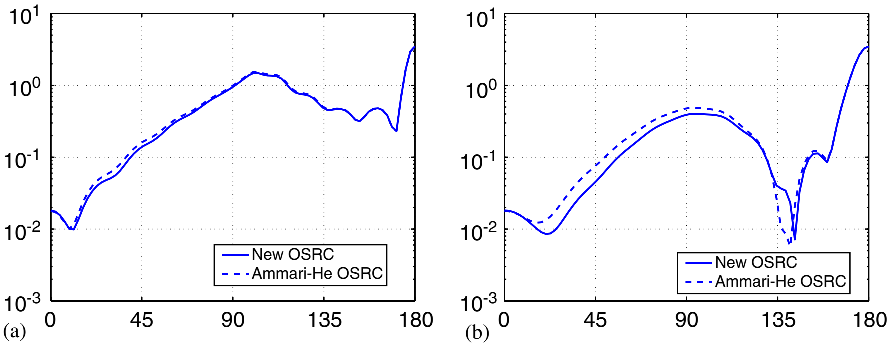

Fig. 8. Relative error in current for a sphere with perfectly conducting boundary conditions for New OSRC (27) and Ammari–He OSRC (24): (a) $\phi = 0°$, $kR = 10$; (b) $\phi = 90°$, $kR = 10$.

different PO approximations produce a smaller error. Since there are large errors in the potential e.g., u_n^{tot} around the shadow boundary, one can improve the backscattering computation by replacing u_n^{tot} with $F \cdot u_n^{\text{tot}}$, where F is a smooth cutoff function that vanishes in the shadow region.

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